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Energy spectrum, potential and inertia functions of a generalized f-oscillator

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Abstract
We consider a generalized four-parameter q-algebra $AA^\dagger - q^\gamma A^\dagger A = q^{\alpha N+\beta}$, $[N, A] = -A$, $[N, A^\dagger] = A^\dagger$, associating with operators A and $A^\dagger$ the nonlinear f-oscillator operators, defined in terms of the usual harmonic oscillator operators as $A \equiv af(N)$ and $A^\dagger \equiv f^*(N)a^\dagger$ (where a and $a^\dagger$ are operators of the Weyl–Heisenberg algebra and $N = a^\dagger a$). The function $f(N)$ is determined from the commutation relations. We write the Hamiltonian for the free f-oscillator and obtain its energy spectrum. Besides, expressing the Hamiltonian in terms of coordinate and momentum, we determine the potential and inertia functions (coordinate-dependent mass) and analyse their behaviour by varying the parameters.

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1. Introduction

The so-called quantum groups, quantum algebras and quantum spaces became the objects of intensive studies in different areas of mathematics and physics after the seminal works in the 1980s [1–5], although deformed commutation relations were considered by several authors earlier [6–10]. Realizations of the deformed algebras associated with the deformed harmonic oscillator ('q-oscillator'), representing deformed boson quanta, were introduced in the seminal papers [11, 12]. Various generalizations and applications of these ideas were considered by a number of authors during the past decade [13–30]. In particular, a sound interpretation of the physical meaning of the q-oscillator as a nonlinear 'f-oscillator', whose frequency depends on the energy (intensity), was given in [31] (relations between anharmonic and q-oscillators were also studied in [32, 33]). Another direction was a search for multiparametric generalizations of the one-parameter q-oscillators [34–36]. A large four-parameter generalized deformed algebra (GDA), encompassing many other constructions as special cases, was proposed in [37].

The aim of our paper is to establish some connections between the concepts of f-oscillators and GDA algebra. We analyse its dynamical nature by writing the Hamiltonian of a free f-oscillator in terms of the usual position and momentum operators P and Q (we use capital letters for the coordinates, in order to distinguish them from the deformation parameters). In this way, we can gain an insight into the Q-dependent potential and effective mass functions arising in the classical limit, and set their dependence on the deformation parameters. The plan of the paper is as follows. In section 2 we give, for the sake of consistency, a brief review of the main concepts related to deformed oscillator algebras. In section 4 we derive relations between q- and f-oscillators and obtain the Hamiltonian and energy spectrum in terms of quantum number operator N. In section 5 we express the Hamiltonian in terms of coordinate and momentum Q and P, up to the quadratic order in P, from which we derive the Q-dependent potential and inertia functions. Finally, section 6 is devoted to a summary, discussions and conclusions.

2. Deformed oscillator algebras

We recall that the algebra associated with a harmonic oscillator (HO) in quantum mechanics (QM), usually called Weyl–Heisenberg algebra, is a three-element (or, generators of a) Lie algebra $\{a, a^\dagger, I\}$ defined by the commutation rules

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = I \qquad [a, I] = [a^\dagger, I] = 0$$

where I is the identity operator, a and $a^\dagger$ are operators that, respectively, destroy and create a single quantum in the Fock basis states

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

and

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \qquad n = 0, 1, 2, 3, \ldots$$

$|0\rangle$ being the vacuum state. The number operator $N = a^\dagger a$ satisfies the commutation relations

$$[N, a,] = -a \qquad [N, a^\dagger] = a^\dagger$$

and its eigenstates are the Fock states

$$N|n\rangle = n|n\rangle.$$

We note that N is not a generator of the Weyl–Heisenberg algebra since it was defined in terms of basic generators. Now, if the relation $N = a^\dagger a$ is not imposed, and instead, N is introduced as a new generator of the algebra, the three-element Lie algebra $\{A, A^\dagger, N\}$ is known as the boson or oscillator algebra, satisfying the same, but now postulated, commutation relations

$$[A, A^\dagger] = I \qquad [N, A] = -A \qquad [N, A^\dagger] = A^\dagger$$

and the identity I commutes with all three generators. This construction goes back to the work by Wigner [38] (who used operators Q and P instead of A and $A^\dagger$), and the related Lie group/algebra, frequently called the oscillator group/algebra, was studied by many authors [39–43].

2.1. Examples of one-parameter deformed oscillator (q-oscillator) algebras

The quantum deformed oscillator (q-oscillator) algebra is the three-element Lie algebra $\{A, A^\dagger, N\}$ plus one-parameter q, which modifies the commutation relations. Even for a single parameter, there exist many different modifications. We list below a few of them.

1. Historically, the first deformed commutation relations were introduced independently by Iwata [6], Arik and Coon [7] and Kuryshkin [8], so this algebra may be called Iwata–Arik-Coon–Kuryshkin (IACK) algebra

$$AA^\dagger - qA^\dagger A = I \qquad [N, A] = -A \qquad [N, A^\dagger] = A^\dagger$$

2. Feinsilver [44] considered the algebra

$$[A, A^\dagger] = q^{-2N} \qquad [N, A] = -A \qquad [N, A^\dagger] = A^\dagger$$

3. One of the most frequently considered examples besides the IACK algebra is the algebra proposed by Biedenharn and Macfarlane [11, 12] (BM algebra)

$$AA^\dagger - qA^\dagger A = q^{-N} \qquad [N, A] = -A \qquad [N, A^\dagger] = A^\dagger \tag{1}$$

whose initial purpose was to generalize Schwinger's boson realization of angular momentum operators on the basis of deformed ('quantum') analogues of algebras $SU(2)$ and $SU(1, 1)$. From (1) the commutation relations

$$[N, A^\dagger A] = 0 \qquad [N, AA^\dagger] = 0$$

follow and a useful relation is

$$A(A^\dagger)^m - (qA^\dagger)^m A = [m](A^\dagger)^{m-1} q^{-N}$$

where

$$[m] \equiv \frac{q^m - q^{-m}}{q - q^{-1}} = \frac{\sinh(m\tau)}{\sinh(\tau)} \qquad q = e^\tau.$$

So, this algebra is symmetric under the interchange $q \leftrightarrow q^{-1}$ or $\tau \leftrightarrow -\tau$. In [23] the terms 'maths' q-bosons and 'physics' q-bosons were proposed for the IACK and BM algebras, respectively.

4. The last example is the so-called 'Tamm–Dancoff algebra' [36, 41, 42, 45]

$$AA^\dagger - qA^\dagger A = q^N \qquad [N, A] = -A \qquad [N, A^\dagger] = A^\dagger.$$

The name is explained by the fact that in this case the energy spectrum is limited from above (see also [46]), resembling the idea of high energy cut-off in the Tamm–Dancoff model of the quantum field theory. For other examples see, e.g., lists in [24, 26, 28].

2.2. Multiparameter q-deformed algebras

A two-parameter quantum algebra $su_{pq}(2)$ was introduced in [34] on the basis of the definition

$$[x]_{pq} = (q^x - p^{-x})/(q - p^{-1}).$$

A similar construction, characterized by the deformations of the form

$$aa^\dagger = q_1^2 a^\dagger a + q_2^{2N} = q_2^2 a^\dagger a + q_1^{2N} \qquad [m] = \left(q_1^{2m} - q_2^{2m}\right)/\left(q_1^2 - q_2^2\right)$$

has been studied in [35] under the name '*Fibonacci oscillator*'.

The authors of [37] proposed the four-parameter GDA

$$AA^{\dagger} - q^{\gamma} A^{\dagger} A = q^{\alpha N+\beta} \qquad [N, A] = -A \qquad [N, A^{\dagger}] = A^{\dagger} \tag{2}$$

which encompasses the above-mentioned algebras. Here α, β and γ are real parameters. Taking $\gamma = 1$ one recovers the three-parameter algebra introduced earlier in [36]. Actually, both algebras become equivalent if one redefines $\tau\gamma = \tau', \alpha/\gamma = \alpha'$ and $\beta/\gamma = \beta'$. The consequences of (2) are the relations

$$\begin{aligned} &A(A^{\dagger})^m - q^{m\gamma}(A^{\dagger})^m = (A^{\dagger})^{m-1} q^{\alpha N+\beta} \frac{q^{m\alpha} - q^{m\gamma}}{q^{\alpha} - q^{\gamma}} \\ &A|n\rangle = \sqrt{F^{\gamma}_{\alpha,\beta}(n; q)}|n-1\rangle \qquad A^{\dagger}|n\rangle = \sqrt{F^{\gamma}_{\alpha,\beta}(n+1; q)}|n+1\rangle \end{aligned} \tag{3}$$

with the basis states

$$|n\rangle = \left[F^{\gamma}_{\alpha,\beta}(n; q)!\right]^{-1/2} (A^{\dagger})^n |0\rangle \qquad n = 1, 2, 3, \ldots \tag{4}$$

where

$$F^{\gamma}_{\alpha,\beta}(n; q) = \begin{cases} q^{\beta} \dfrac{q^{n\alpha} - q^{n\gamma}}{q^{\alpha} - q^{\gamma}} & \text{for} \quad \alpha \neq \gamma \\ n q^{\beta+\gamma(n-1)} & \text{for} \quad \alpha = \gamma \end{cases} \tag{5}$$

and

$$F^{\gamma}_{\alpha,\beta}(n; q)! \equiv F^{\gamma}_{\alpha,\beta}(n; q) F^{\gamma}_{\alpha,\beta}(n-1; q) \cdots F^{\gamma}_{\alpha,\beta}(2; q) F^{\gamma}_{\alpha,\beta}(1; q). \tag{6}$$

3. The f-oscillators

The authors of [31, 47] introduced a realization for the operator A and its adjoint $A^{\dagger}$ in terms of the so-called f-oscillators, defined as a nonlinear expansion of the usual harmonic oscillator operators a and $a^{\dagger}$

$$A \equiv a f(N) \qquad A^{\dagger} \equiv f^{*}(N) a^{\dagger} \qquad \text{and} \qquad N \equiv a^{\dagger} a \tag{7}$$

however, $N_d = A^{\dagger} A \neq N$. Such realizations were known earlier [10, 13, 19], but the authors of [31] gave an explicit physical interpretation of A as the operator describing an anharmonic oscillator with intensity dependent frequency. The function $f(N)$ is specific to each q-deformed algebra, and since herein we are going to deal with the GDA, this function will also depend on the four parameters, q (or τ), α, β and γ.

A commutation relation may be established,

$$[A, A^{\dagger}] = \phi(N) \tag{8}$$

where the RHS is a function of N,

$$\phi(N) = |f(N+1)|^2 (N+1) - |f(N)|^2 N \tag{9}$$

thus, each specific function $f(N)$ implies a particular commutation relation. By its turn, the Heisenberg equation of motion for A (or $A^{\dagger}$) $\dot{A} + \mathrm{i}[A, H(A, A^{\dagger}, N)] = 0$ will depend on both, the particular Hamiltonian $H(A, A^{\dagger}, N)$ and the commutation relation (8) (or $f(N)$). For $H(A, A^{\dagger}, N) = H(N)$, the dynamical equation for A is

$$\dot{A} + \mathrm{i}\omega_{+}(N) A = 0 \tag{10}$$

and the nonlinear frequency $\omega_{+}(N)$ is defined from

$$[A, H(N)] = \omega_{+}(N) A \tag{11}$$

according to the commutation relations (2), $H(N)$ and $\omega_+(N)$ are related by

$$H(N+1) - H(N) = \omega_+(N). \tag{12}$$

Considering the Fourier transform (FT) for $H(N)$ and $\omega_+(N)$,

$$\tilde{H}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \qquad \tilde{\omega}_+(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega_+(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \tag{13}$$

there is a direct relation between frequency and Hamiltonian,

$$H(N) = \int_{-\infty}^{\infty} \frac{\tilde{\omega}_+(k)}{\mathrm{e}^{\mathrm{i}k} - 1}\,\mathrm{e}^{\mathrm{i}kN}\,\mathrm{d}k \tag{14}$$

or

$$\omega_+(N) = \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}k(N+1)} - \mathrm{e}^{\mathrm{i}kN})\tilde{H}(k)\,\mathrm{d}k. \tag{15}$$

As an example let us consider the Hamiltonian for the free f-oscillator

$$H(N) = \frac{\omega_0}{2}(A^\dagger A + AA^\dagger) = \frac{\omega_0}{2}[|f(N+1)|^2(N+1) + |f(N)|^2 N] \tag{16}$$

where N is a constant of motion, and the frequency can also be expressed as

$$\omega_+(N) = \frac{\omega_0}{2}[\phi(N+1) + \phi(N)] \tag{17}$$

$$= \frac{\omega_0}{2}[|f(N+2)|^2(N+2) - |f(N)|^2 N]. \tag{18}$$

The solution to (10) can be written as

$$A(t) = \mathrm{e}^{-\mathrm{i}\omega_+(N)(t-t_0)}A(t_0) \tag{19}$$

or in terms of the evolution operator $U(t) = \mathrm{e}^{-\mathrm{i}H(N)t}$,

$$A(t) = \mathrm{e}^{\mathrm{i}H(N)(t-t_0)}A(t_0)\,\mathrm{e}^{-\mathrm{i}H(N)(t-t_0)} = \mathrm{e}^{-\mathrm{i}(H(N+1)-H(N))(t-t_0)}A(t_0). \tag{20}$$

We could also have written

$$A(t) = A(t_0)\,\mathrm{e}^{-\mathrm{i}\omega_-(N)(t-t_0)} \tag{21}$$

where

$$\begin{aligned}\omega_-(N) &= \frac{\omega_0}{2}[\phi(N) + \phi(N-1)] = [H(N) - H(N_1)] \\ &= \frac{\omega_0}{2}[|f(N+1)|^2(N+1) - |f(N-1)|^2(N-1)].\end{aligned} \tag{22}$$

The definition for the frequency is ambiguous, because, since it is a function of N, it will depend on how the equation of motion is written: as $\dot{A} + \mathrm{i}\omega_+(N)A = 0$ or as $\dot{A} + \mathrm{i}A\omega_-(N) = 0$. It is worth noting that the operator a shares the same frequency associated with A since the equation of motion is the same, $\dot{a} + \mathrm{i}\omega_+(N)a = 0$ (or $\dot{a} + \mathrm{i}a\omega_-(N) = 0$).

Related to the above discussion, it was shown in [47] that for quantum systems the vector field associated with the equations of motion may admit alternative Hamiltonian descriptions, both in the Schrödinger and Heisenberg pictures. An equation of motion does not define uniquely the quantum commutation relations, which is known as Wigner's problem. For instance if one considers the equation of motion for the linear oscillator amplitude operator a,

$$\dot{a} + \mathrm{i}\omega_0 a = 0 \tag{23}$$

one can verify that the commutation relation $[a, a^\dagger] = 1$ and the Hamiltonian $H_0 = \omega_0(a^\dagger a + 1/2)$ are compatible. Now, since the number operator $N = a^\dagger a$ is a constant of motion, defining the nonlinear amplitude operator $A = af(N)$, $f(N)$ being an invertible function, one gets the same equation of motion for A,

$$\dot{A} + \mathrm{i}\omega_0 A = 0 \tag{24}$$

however, the commutation relation changes to (8) and (9), and the Hamiltonian still satisfies $[A, H_0(N)] = \omega_0 A$.

In the next section we derive relations (3)–(5) using the f-oscillator map.

4. A nonlinear realization for the GDA *q*-algebra

Considering the definition (7), it is trivial to verify that

$$af(N) = f(N+1)a \quad \text{and} \quad f^*(N)a^\dagger = a^\dagger f^*(N+1). \tag{25}$$

Introducing relations (25) into the algebra (2), one obtains the equation

$$(N+1)|f(N+1)|^2 - q^\gamma N|f(N)|^2 = q^{\alpha N+\beta}. \tag{26}$$

Making the substitution $f(N) = q^{\alpha N/2}h(N)/\sqrt{N}$,

$$q^{\alpha-\beta}|h(N+1)|^2 - q^{\gamma-\beta}|h(N)|^2 = 1 \tag{27}$$

which suggests looking for a solution in the form

$$|h(N)|^2 = A_1(q) + A_2(q)\,\mathrm{e}^{u(q)N}. \tag{28}$$

For a given function $A_2(q) \neq 0$, the functions $A_1(q)$ and $u(q)$ can be found by substituting (28) into (27) and equating terms with the same powers, e^0 and $\mathrm{e}^{u(q)N}$. Thus we find $A_1(q) = q^\beta/(q^\alpha - q^\gamma)$ and $u(q) = \ln q^{\gamma-\alpha}$. Therefore,

$$|f(N)|^2 = \frac{1}{N}\left(\frac{q^{\beta+\alpha N}}{q^\alpha - q^\gamma} + A_2(q)q^{\gamma N}\right).$$

In order to avoid a singularity in $f(N)$ for $\alpha = \gamma$, we set $A_2(q) = -q^\beta/(q^\alpha - q^\gamma)$. Thus

$$|f(N)|^2 = \begin{cases} \dfrac{q^\beta}{N}\dfrac{q^{\alpha N} - q^{\gamma N}}{q^\alpha - q^\gamma} & \text{for} \quad \alpha \neq \gamma \\ q^{\beta+\gamma(N-1)} & \text{for} \quad \alpha = \gamma. \end{cases} \tag{29}$$

The right-hand side of (29) is positive, if all parameters q, α, β and γ are real (and, moreover, q is positive). In this case one can choose (suppressing an unessential phase) $f(N) = \sqrt{|f(N)|^2}$. The q-deformed Fock states obey the relations (3) and (4) with $F(n;q) = n|f(n;q)|^2$ (we use capital letter N for the photon number operator and lowercase letter n for its eigenvalues).

It is convenient to introduce new deformation parameters according to relations

$$q = \mathrm{e}^\tau \qquad \alpha = \rho + \mu \qquad \gamma = \rho - \mu.$$

Then

$$|f(N)|^2 = \frac{\sinh(\tau\mu N)}{N\sinh(\tau\mu)}\exp\{\tau[\beta + \rho(N-1)]\} \tag{30}$$

while equation (9) becomes

$$\phi(N) = \frac{\mathrm{e}^{\tau\beta+\tau\rho(N-1)}[\mathrm{e}^{\tau\rho}\sinh(\tau\mu(N+1)) - \sinh(\tau\mu N)]}{\sinh(\tau\mu)}. \tag{31}$$

The Hamiltonian of the free f-oscillator (16) can be written explicitly as

$$H = \frac{\hbar\omega_0}{2}\,\mathrm{e}^{\tau(\beta+\rho N)}\left\{\frac{\sinh(\tau\mu[N+1])}{\sinh(\tau\mu)} + \mathrm{e}^{-\tau\rho}\frac{\sinh(\tau\mu N)}{\sinh(\tau\mu)}\right\}. \tag{32}$$

The eigenvalues E_n are obtained by replacing the operator N in (32) by integers $n = 0, 1, 2, \ldots,$ they remain unchanged by changing the sign of μ, while they do not have a definite symmetry under the change of the sign of ρ. Obviously, $\lim_{\tau\to 0} E_n = \hbar\omega(n + 1/2)$. The number of (discrete) energy levels is infinite, and the asymptotical behaviour of the spectrum for $n \to \infty$ and $\tau \neq 0$ is governed by the exponential factor $\exp[\tau n(\rho+|\mu|)]$. If $\tau(\rho+|\mu|) > 0$ the energy grows unlimitedly with the increase of n. In contradistinction, if $\rho+|\mu| = 0$, i.e. either $\alpha = 0$ or $\gamma = 0$, then E_n tends monotonically to an upper bound that depends on the parameters,

$$E_{\max} = \frac{\hbar\omega_0}{2}\frac{\exp[\tau(\beta-\rho)]}{\sinh(\tau|\mu|)}. \tag{33}$$

When $\tau(\rho + |\mu|) < 0$, the energy spectrum exhibits an initial increase, but with growing n it attains some maximal value and then goes to zero for $n \to \infty$. In the special case $\mu = 0$ (i.e. $\alpha = \gamma$) and $\beta = 0$ we have

$$E_n = \frac{\hbar\omega_0}{2}\,\mathrm{e}^{\tau\rho n}[1 + n(1 + \mathrm{e}^{-\tau\rho})]$$

and the frequency is

$$\omega_+(n) = \omega_0\,\mathrm{e}^{\tau\rho n}[\mathrm{e}^{\tau\rho} + n\sinh(\tau\rho)]. \tag{34}$$

We can gain a better insight into the effects of deformation by assuming $\tau = 1$, $\rho n \ll 1$, and keeping terms up to n^2, in such case we have

$$E_n \approx \frac{\hbar\omega_0}{2}[1 + n(1 + \rho + \mathrm{e}^{-\rho}) + n^2\rho(1 + \mathrm{e}^{-\rho}] \tag{35}$$

which is characteristic of a Kerr medium spectrum, and

$$\omega_+(n) \approx \omega_0[\mathrm{e}^{\rho} + n\rho(1 + \mathrm{e}^{\rho}) + n^2\rho^3] \tag{36}$$

so, even for a small nonlinearity energy and frequency are not proportional.

4.1. An example: the Kerr medium

As a practical physical example from optics we consider a Kerr medium, where the monochromatic field Hamiltonian contains, to lowest order, a nonlinear term proportional to $N(N-1)$ [48, 49],

$$H_{\mathrm{kerr}}(N) = \frac{\hbar\omega_0}{2}(2N + 1) + \frac{\kappa}{2}N(N-1). \tag{37}$$

Assuming small values of ρ and μ^2 (which is the lowest order in μ) and expanding the Hamiltonian (16) we get

$$\begin{aligned} H_N &= \frac{\hbar\omega_0}{2}\left[(2N+1) + \frac{1}{6}\mu^2 N + \left(\frac{1}{2}\mu^2 + 2\rho\right)N^2 + O(\rho^2, \rho\mu^2, \mu^4)\right] \\ &\approx \frac{\hbar\omega_0}{2}\left[(2N+1) + \left(\frac{2}{3}\mu^2 + 2\rho\right)N + \left(\frac{1}{2}\mu^2 + 2\rho\right)N(N-1)\right]. \end{aligned} \tag{38}$$

Since the nonlinear term in Hamiltonian (37) contains only one parameter, the Hamiltonian (38) reproduces (37) by setting $\mu^2 = -3\rho$ and $\rho = 2\kappa/\omega_0$. Thus, the Kerr medium transforms a usual linear harmonic oscillator into an f-oscillator.

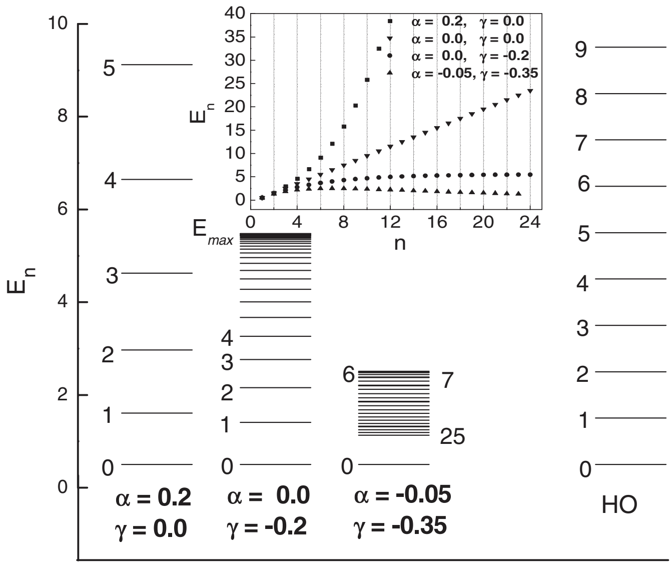

Figure 1. Energy levels for Hamiltonian (32) for several values of the parameters α and γ. One sees dilation and compression of the spectrum, compared to the harmonic oscillator, on the right. In the inserted figure one has the energies on the same scale.

Alternatively, we could have set, *ab initio*, in (16)

$$f(N) = \left[1 + \frac{\kappa(N-1)^2}{2\omega_0 N}\right]^{1/2} \tag{39}$$

to obtain (37). So, the parameters of the GDA are related to κ, which is proportional to the nonlinear susceptibility parameter.

4.2. The energy spectrum

In figure 1 we display the spectra for four sets of values for (α, γ): $(0.0, 0.0)$ stands for the HO, the energy levels are equally spaced. The other sets give nonlinear spectra: for $(0.2, 0.0)$ the spectrum suffers a dilation, the gaps between successive energy levels increase with n. For $(0.0, -0.2)$ the energy spectrum is compact, the gaps between successive energy levels decrease and as $n \to \infty$ eventually go to zero, with upper-bound energy (33). For $(-0.05, -0.35)$, besides being compact, the energy spectrum also bends, the energy attains a maximum value for $n = 6$ then decreases for increasing n. When $\alpha, \gamma < 0$, the higher energy level occurs for the positive integer n that is nearest to

$$\bar{n} = \frac{1}{\alpha - \gamma} \ln\left(\frac{1 + \mathrm{e}^{\gamma}}{1 + \mathrm{e}^{\alpha}} \cdot \frac{\gamma}{\alpha}\right) \tag{40}$$

obtained from $\mathrm{d}E_n/\mathrm{d}n = 0$ and where,

$$\left.\frac{\mathrm{d}^2 E_n}{\mathrm{d}n^2}\right|_{n=\bar{n}} = -\alpha\gamma E_{\bar{n}}.$$

We note that the right-hand sides of (29) and (30) remain real, even if the parameter μ becomes pure imaginary (so that $\alpha = \gamma^*$). However, in such a case μ cannot assume arbitrary

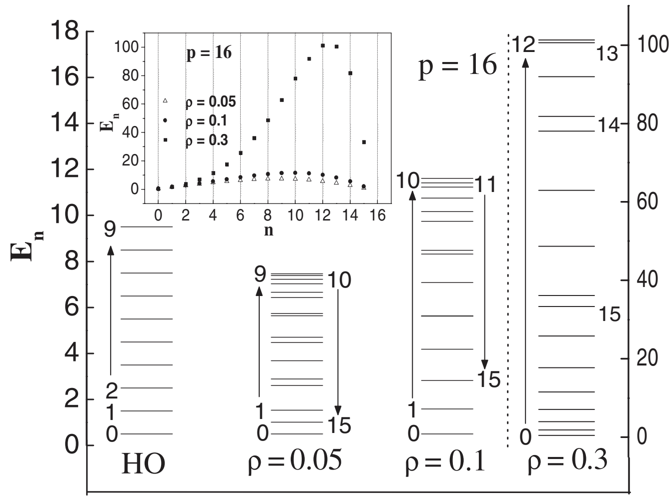

Figure 2. Energy levels for Hamiltonian (42) with $p = 16$ for several values of ρ. One sees dilation and compression of the spectrum, compared to the harmonic oscillator, on the left. For $\rho = 0.3$ the scale is different. In the inserted figure all energies are on the same scale.

values. In order to ensure positiveness of $|f(N)|^2$ we must set $\tau\mu = \mathrm{i}\pi/p$ with p being a positive integer. Thus we arrive at the function

$$|f_p(N;\beta,\rho,\tau)|^2 = \frac{\sin(\pi N/p)}{N\sin(\pi/p)}\exp\{\tau[\beta+\rho(N-1)]\} \tag{41}$$

which results in a *truncated* deformed Fock states basis, having only p states $|0\rangle, |1\rangle, \ldots |p-1\rangle$, since $f_p(p;\beta,\rho,\tau) = 0$. The Hamiltonian (32) assumes the form

$$H_p = \frac{\hbar\omega}{2}\,\mathrm{e}^{\tau(\beta+\rho N)}\left\{\frac{\sin(\pi[N+1]/p)}{\sin(\pi/p)} + \mathrm{e}^{-\tau\rho}\frac{\sin(\pi N/p)}{\sin(\pi/p)}\right\}. \tag{42}$$

Since there is one Hamiltonian for each integer value of p, we call H_p a class p Hamiltonian, or p-Hamiltonian class. Once, in all cases, the parameter β enters as a mere scaling factor, $\mathrm{e}^{\tau\beta}$, we set $\beta = 0$ in the further analysis, since interesting physics arises from the interplay between parameters α and γ.

As in figure 1, in figure 2 we display the energy spectra for $p = 16$ and $\rho = 0.05, 0.1, 0.3$, respectively. On the left side we give the HO spectrum for the sake of comparison, and for $\rho = 0.3$ we note that the energy spectrum is on a different scale. Spacing regularity between the energy levels $E_{p,n}$ is lost and all spectra are compact; they bend after attaining a maximum value (the sequences of the energies can better be viewed in the insertion) because

$$\left.\frac{\mathrm{d}^2E_{p,n}}{\mathrm{d}n^2}\right|_{n=\bar{n}} = -\left[\left(\frac{\pi}{p}\right)^2 + \rho^2\right]E_{p,\bar{n}}.$$

The point of maximum $\bar{n}$ is determined from $\mathrm{d}E_{p,n}/\mathrm{d}n = 0$, being,

$$\bar{n} = \frac{p}{\pi}\left\{\pi + \arctan\left[\frac{(\pi/p\rho)(\cos(\pi/p)+\mathrm{e}^{-\rho})+\sin(\pi/p)}{(\pi/p\rho)\sin(\pi/p)-(\cos(\pi/p)+\mathrm{e}^{-\rho})}\right]\right\} \tag{43}$$

the value of n for the highest energy level $E_{p,n}$, in any spectrum, will be the nearest positive integer to $\bar{n}$.

5. Potential and mass functions

Once we have obtained the Hamiltonians and energy levels, we shall try to understand the meaning of the energy spectra behaviour by looking at the classical limit of the Hamiltonians, i.e. the Hamiltonians expressed in terms of position and momentum operators, P and Q. More precisely, we will determine the potential energy and inertia function (or the effective mass) in terms of Q. Assuming small values for the parameters and keeping terms up to the lowest orders with respect to the momentum P in (32) and (42), we can write them in the form

$$H(P,Q) = V(Q) + \tfrac{1}{2} P M^{-1}(Q) P + P^2 W(Q) P^2 + \cdots. \tag{44}$$

The symmetric quadratic form $Q^2 + P^2$, present in the exact Hamiltonians, is broken whenever the expansion (44) is, at any order, truncated. Recalling definition (7), we set, as usual, the non-Hermitian operators

$$a = (\sqrt{m\omega}\tilde{Q} + \mathrm{i}\tilde{P}/\sqrt{m\omega})/\sqrt{2\hbar} \qquad a^\dagger = (\sqrt{m\omega}\tilde{Q} - \mathrm{i}\tilde{P}/\sqrt{m\omega})/\sqrt{2\hbar}$$

thus

$$N = a^\dagger a = \frac{1}{\hbar\omega}\left(\frac{m\omega^2}{2}\tilde{Q}^2 + \frac{1}{2m}\tilde{P}^2\right) - \frac{1}{2} = \frac{1}{2}(Q^2 + P^2 - 1) \tag{45}$$

where $Q = (m\omega/\hbar)^{1/2}\tilde{Q}$ and $P = (m\omega\hbar)^{-1/2}\tilde{P}$ are dimensionless conjugated coordinate and momentum variables.

Since the Hamiltonians (32) and (42) contain terms such as $\exp(xN)$, x being a real or an imaginary dimensionless c-number, we write $\exp(xN) = \exp(-x/2)\exp(aQ^2 + bP^2)$ with $a = b = x/2$. Therefore, up to linear terms in bP^2, we have (see appendix A)

$$L_b\{\mathrm{e}^{xN}\} = \mathrm{e}^{-x/2} L_b\{\exp[x(Q^2 + P^2)/2]\} = \mathrm{e}^{-x/2}[P u_1(Q) P + u_2(Q)] \tag{46}$$

where

$$u_1(Q;x) = \frac{x}{2}\,\mathrm{e}^{xQ^2/2} \qquad u_2(Q;x) = \left[1 - \frac{x^2}{4}\left(1 + \frac{2x}{3}Q^2\right)\right]\mathrm{e}^{xQ^2/2} \tag{47}$$

and the symbol $L_b\{f(b)\}$ means: keep only the terms of order b^0 and b^1 in the expansion of function $f(b)$.

5.1. Continuous parameters: infinite set of countable energy levels

In calculating the first two terms in (44), the use of the $\alpha - \gamma$ parametrization leads to a much simpler form than the one that uses the parameters ρ, μ. Moreover, since the parameter τ in (32) appears always multiplying α and γ (or ρ and μ), we consider it to be absorbed by them, i.e. hereafter we make the substitutions $\tau\alpha \to \alpha$, $\tau\gamma \to \gamma$. To write the Hamiltonian (32) up to quadratic terms in P, we have to express the hyperbolic functions in exponential form and then calculate the 'b-linear' terms of the combination $(1+\mathrm{e}^{\alpha})L_b\{\mathrm{e}^{\alpha N}\} - (1+\mathrm{e}^{\gamma})L_b\{\mathrm{e}^{\gamma N}\}$. Using equations (46) and (47), after some algebra, we get the following inverse effective mass and potential functions:

$$M^{-1}(Q;\alpha,\gamma) = (\mathrm{e}^{\alpha} - \mathrm{e}^{\gamma})^{-1}\left[\alpha\,\mathrm{e}^{\alpha Q^2/2}\cosh\frac{\alpha}{2} - \gamma\,\mathrm{e}^{\gamma Q^2/2}\cosh\frac{\gamma}{2}\right] \tag{48}$$

$$\begin{aligned} V(Q;\alpha,\gamma) = (\mathrm{e}^{\alpha} - \mathrm{e}^{\gamma})^{-1}\Bigg\{&\left[1 - \frac{\alpha^2}{4}\left(1 + \frac{2\alpha}{3}Q^2\right)\right]\mathrm{e}^{\alpha Q^2/2}\cosh\frac{\alpha}{2} \\ &- \left[1 - \frac{\gamma^2}{4}\left(1 + \frac{2\gamma}{3}Q^2\right)\right]\mathrm{e}^{\gamma Q^2/2}\cosh\frac{\gamma}{2}\Bigg\}. \end{aligned} \tag{49}$$

In the case of $\alpha = \gamma$ (or $\mu = 0$) we obtain

$$M^{-1}(Q;\gamma,\gamma) = \mathrm{e}^{-\gamma}\left[\frac{\gamma}{2}\sinh\frac{\gamma}{2} + \cosh\frac{\gamma}{2} + \frac{\gamma Q^2}{2}\cosh\frac{\gamma}{2}\right]\mathrm{e}^{\gamma Q^2/2} \tag{50}$$

$$V(Q;\gamma,\gamma) = \frac{\mathrm{e}^{-\gamma}}{2}\left\{\left[\left(1-\frac{\gamma^2}{4}\right)\sinh\frac{\gamma}{2} - \gamma\cosh\frac{\gamma}{2}\right] + \left[\left(1-\frac{5}{4}\gamma^2\right)\cosh\frac{\gamma}{2} - \frac{\gamma^3}{6}\sinh\frac{\gamma}{2}\right]Q^2 - \frac{\gamma^3}{6}Q^4\cosh\frac{\gamma}{2}\right\}\mathrm{e}^{\gamma Q^2/2}. \tag{51}$$

For $\alpha = \gamma = 0$, $M(Q;0,0) = 1$ and $V(Q;0,0) = Q^2/2$, as it should. Interestingly, a quartic term in Q is additionally present in (51) while it is absent in (49).

5.2. p-Hamiltonian class: finite number of energy levels

In the case of Hamiltonians of class p, equation (42), calculations similar to those performed in the previous subsection furnish the following expressions:

$$M_p^{-1}(Q;\rho) = \frac{\exp[\rho(Q^2-1)/2]}{2\sin(\pi/p)}\left\{\rho\left[\sin\left(\frac{\pi}{2p}(1+Q^2)\right) - \mathrm{e}^{-\rho}\sin\left(\frac{\pi}{2p}(1-Q^2)\right)\right] + \frac{\pi}{p}\left[\cos\left(\frac{\pi}{2p}(1+Q^2)\right) + \mathrm{e}^{-\rho}\cos\left(\frac{\pi}{2p}(1-Q^2)\right)\right]\right\} \tag{52}$$

$$\begin{aligned}V_p(Q;\rho) = \frac{\exp[\rho(Q^2-1)/2]}{2\sin(\pi/p)}&\left\{\left[\left(1-\frac{\rho^2}{4}+\frac{\pi^2}{4p^2}\right) - \rho\left(\frac{\rho^2}{6}-\frac{\pi^2}{p^2}\right)Q^2\right]\right.\\ &\times\left[\sin\left(\frac{\pi}{2p}(1+Q^2)\right) - \mathrm{e}^{-\rho}\sin\left(\frac{\pi}{2p}(1-Q^2)\right)\right]\\ &-\frac{\pi}{2p}\left[\rho + \left(\rho^2 - \frac{2\pi^2}{p^2}\right)Q^2\right]\left[\cos\left(\frac{\pi}{2p}(1+Q^2)\right)\right.\\ &\left.\left.+\mathrm{e}^{-\rho}\cos\left(\frac{\pi}{2p}(1-Q^2)\right)\right]\right\}.\end{aligned} \tag{53}$$

For $\rho = 0$ and $1 \leqslant p < \infty$, equations (52) and (53) go to

$$M_p^{-1}(Q;0) = \frac{\pi/(2p)}{\sin[\pi/(2p)]}\cos\left(\frac{\pi}{2p}Q^2\right) \tag{54}$$

$$V_p(Q;0) = \frac{1}{2\sin[\pi/(2p)]}\left\{\left(1+\frac{\pi^2}{4p^2}\right)\sin\left(\frac{\pi Q^2}{2p}\right) + \left(\frac{\pi}{p}\right)^3 Q^2\cos\left(\frac{\pi Q^2}{2p}\right)\right\}. \tag{55}$$

Due to equation (54), the effective mass becomes infinite at points $Q = \pm\sqrt{p}$ and the particle cannot go through these points (in the quasi-classical approximation), so, the value of the modulus of the classical coordinate Q must be restricted within the open interval $(-\sqrt{p}, \sqrt{p})$. Finiteness of parameter p implies an infinite mass at some spatial location, and because of this infinite inertia the particle must have its motion confined to a spatial region, where the mass function is non-negative. Under this restriction the potential function is also spatially limited, oscillations do not show up, in spite of the presence of sin and cos functions of Q^2.

For $\rho \neq 0$ and $p \to \infty$, one gets

$$M_\infty^{-1}(Q;\rho) = \frac{1}{2}\exp[\rho(Q^2-1)/2]\left[\frac{\rho}{2}(1-\mathrm{e}^{-\rho}) + (1+\mathrm{e}^{-\rho})\left(1+\frac{\rho}{2}Q^2\right)\right] \tag{56}$$

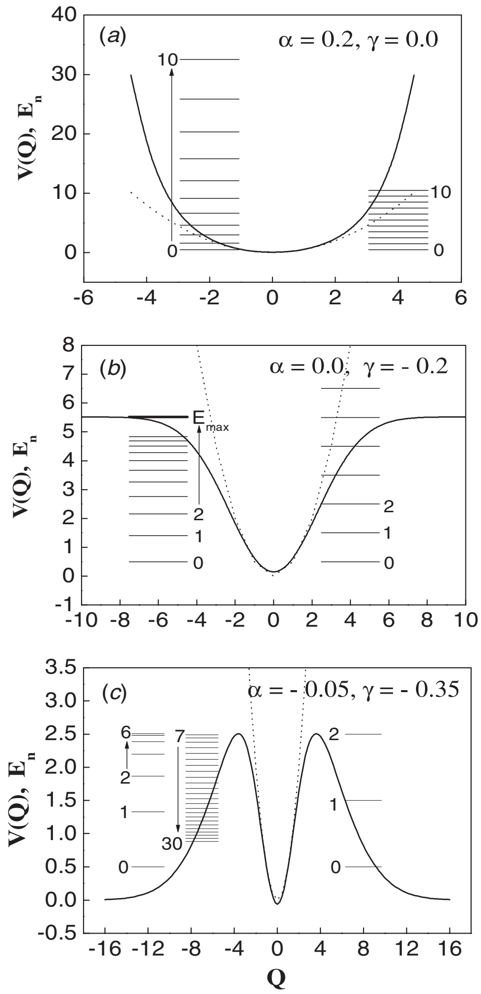

Figure 3. The potential energy in Hamiltonian (32) is plotted (solid line) as a function of the variable Q for several values of the parameters α and γ. For the sake of comparison, the harmonic oscillator potential is plotted (dashed line) and energy levels are added.

$$V_\infty(Q;\rho) = \frac{1}{4}\exp[\rho(Q^2-1)/2]\left\{\left(1-\frac{\rho^2}{4}-\frac{\rho^3}{6}Q^2\right)(1-\mathrm{e}^{-\rho}) - \left[\rho-\left(1-\frac{5}{4}\rho^2\right)Q^2+\frac{\rho^3}{6}Q^4\right](1+\mathrm{e}^{-\rho})\right\} \tag{57}$$

and the allowed range for Q becomes unrestricted. For $\rho = 0, M_\infty(Q;0) = 1$ and $V_\infty(Q;0) = Q^2/2$, as expected.

5.3. Discussion of figures

In figures 3(*a*)–(*c*) we plotted the deformed potential $V(Q)$ (solid line) for three values of parameters (α,γ): (*a*) (0.2, 0.0), (*b*) (0.0, −0.2), (*c*) (−0.05, −0.35). The dashed lines represent the HO potential. On the left side of each figure we set a few energy levels of the

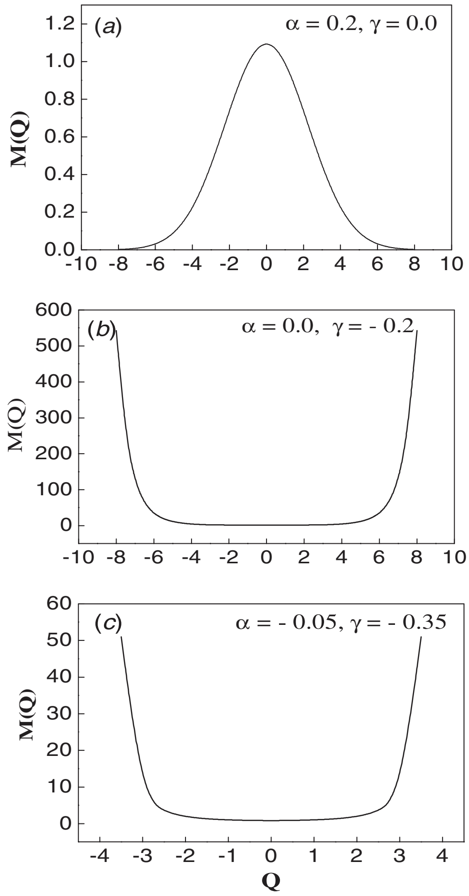

Figure 4. The effective mass function for Hamiltonian (32) as a function of Q for several values of the parameters α and γ.

exact spectrum, while on the right we put, for the sake of comparison, a few levels of the HO energy spectrum. In figures 4(*a*)–(*c*) the effective mass functions are plotted; it is worth recalling that for the HO, $M(Q) = 1$. Three different sets of small (by small we mean their moduli are less than 1) parameters show quite different physical situations.

In figure 3(*a*) the potential shows at $Q = 0$ a smaller curvature than that for the HO and it is unbounded (its value increases indefinitely with higher values of $|Q|$) like the HO. For the mass function, in figure 4(*a*), while it is slightly larger than 1 at $Q = 0$, it has a bell shape, going to 0 asymptotically. These features are compatible with the dilated spectrum (compared with the HO), where the gaps between succeeding energy levels increase with increasing n.

In figure 3(*b*) the potential looks like a shallow dip, whose curvature, at $Q = 0$, is larger than that of the HO; it goes to some asymptotic value as $|Q| \to \infty$. The mass function in figure 4(*b*) is nearly equal to 1 around $Q = 0$, its value increases smoothly with increasing $|Q|$ (note that the mass scale is much larger than that shown in figure 3(*a*)), then it shows a sudden sharp increase near the points where $M(Q_\infty) \to \infty$. The particle inertia increases dramatically, so, although having an infinite number of energy levels, the spectrum is bounded, and its upper-bound energy is given in equation (33). On the right we can see a few energy levels of the HO.

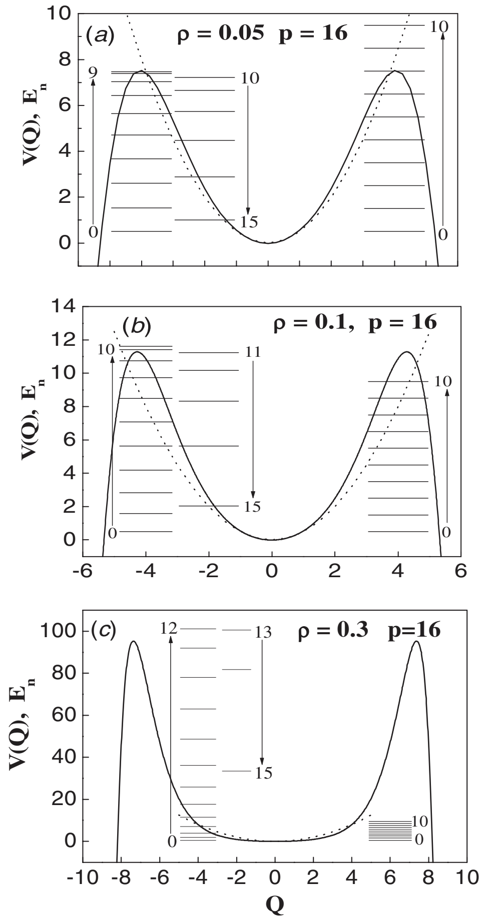

Figure 5. The potential energy in Hamiltonian (42) with $p = 16$ is plotted (solid line) as a function of Q for several values of ρ. For the sake of comparison, the harmonic oscillator potential is plotted (dashed line) and energy levels are added.

In figure 3(*c*), the potential shows a double-hump shape, with larger curvature, at $Q = 0$, than that of the HO; as $|Q| \to \infty$ the potential goes to zero. As in the case of figure 4(*b*), in figure 4(*c*) the mass goes to infinity at the points where the potential attains its maximum value. This behaviour is reflected in the energy spectrum: for small values of n the spectrum is compressed in its spacings, the energy levels attain a maximum value, quite below the maximum of the potential and then decrease monotonically with increasing n. This fact is due to the particle inertia: with increasing n, the mass tends to increase and the particle motion is almost halted, so, its energy decreases. The motion is confined inside the well, although the particle can tunnel through the barriers.

In figures 5(*a*)–(*c*), we plotted the potential energy function and energy levels for a p-Hamiltonian, with $p = 16$ and verified that for (*a*) $\rho = 0.05$, (*b*) $\rho = 0.1$ and (*c*) $\rho = 0.3$, all potentials (solid lines) present a double-hump shape, and assume negative values beyond the points where $V(Q) = 0$. The dashed lines represent the quadratic (HO) potential. In all three cases the energy spectra bend, after attaining a maximum value at an integer n that is nearest to (43), which is due to the increasing inertia, as shown in figures 6(*a*)–(*c*), where we

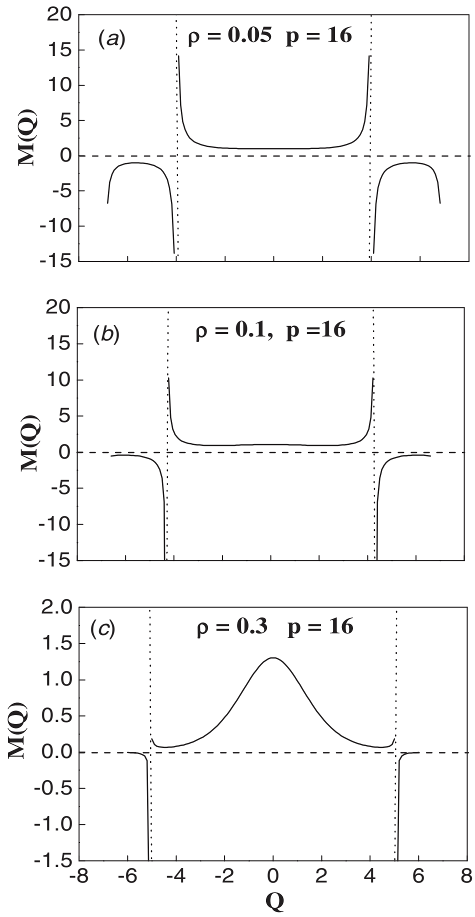

Figure 6. The effective mass function for Hamiltonian (42) with $p = 16$ as a function of Q for several values of ρ.

plotted the effective masses. It is worth noting that beyond the points $\pm Q_\infty$, where

$$Q_\infty^2 = \frac{2p}{\pi}\left\{\pi - \arctan\left[\frac{\rho(1-\mathrm{e}^{-\rho})\sin(\pi/(2p)) + (\pi/p)(1+\mathrm{e}^{-\rho})\cos(\pi/(2p))}{\rho(1+\mathrm{e}^{-\rho})\cos(\pi/(2p)) - (\pi/p)(1-\mathrm{e}^{-\rho})\sin(\pi/(2p))}\right]\right\}$$

(at these points we have $M(\pm Q_\infty) = \infty$), the effective mass $M(Q)$ assumes negative values.

6. Summary and conclusions

We have considered the GDA, as proposed in [36, 37], and associated the abstract operators with the so-called f-oscillator operators, related to the usual operators of the Weyl–Heisenberg group. We wrote the Hamiltonian of a presumed particle represented by the GDA for two possible parametrizations: (a) all parameters real or (b) complex valued parameters, thus obtaining two kinds of Hamiltonians whose energy spectra are nonlinear in the quantum number n.

For case (a) we verified that by slightly changing the parameter values, the energy spectra suffer dramatic changes, getting unbounded dilated (in comparison to the HO equidistant levels) spectrum to compact or bounded spectrum, linear or bent. An insight into this behaviour was obtained by looking at the classical correspondence of the Hamiltonian. For small parameter limit we obtained the potential energy and effective mass functions showing very different behaviour by a slight change of the parameters. Potentials may be concave as the HO, although with different curvature, or showing a double-hump. The effective masses, depending on the position coordinate, may be limited or unlimited, although always being positive. For case (b) all p-Hamiltonians show compact and bent spectra, the bending occurring when the mass may go to infinity at some points of space. All potentials present a double-hump behaviour, although we believe that tunnelling is forbidden due to singularity in the masses. Since negative masses are not ruled out, particles of opposite mass signs should repel each other. Actually, a deeper analysis of particle confinement is needed to consider the wavefunction behaviour. It is largely known that the complex spectra of atomic nuclei are attributed to many kinds of motion, such as rotational, vibrational and other collective modes, so, we are conjecturing that the nonlinear f-oscillator may play an important role in explaining energy levels in the nuclei spectra. The calculations are currently being performed and the results will be presented at due course.

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Appendix A. Expanding exp$[aQ^2 + bP^2]$

Since in

$$\exp(aQ^2 + bP^2) = \sum_{n=0}^{\infty} \frac{1}{n!}(aQ^2 + bP^2)^n \tag{A.1}$$

we can rewrite the binomial factor as a power series in bP^2,

$$\begin{aligned}(aQ^2 + bP^2)^n &= (aQ^2)^n + \sum_{l,m=0}^{n-1} (aQ^2)^m (bP^2)(aQ^2)^l \delta_{m+l,n-1} \\ &\quad + \sum_{k,l,m=0}^{n-2} (aQ^2)^m (bP^2)(aQ^2)^l (bP^2)(aQ^2)^k \delta_{m+l+k,n-2} + \cdots\end{aligned}$$

so (A.1) becomes

$$\mathrm{e}^{aQ^2+bP^2} = \mathrm{e}^{aQ^2} + b \sum_{n=1}^{\infty} \frac{a^{n-1}}{n!} \sum_{m=0}^{n-1} Q^{2m} P^2 Q^{2(n-m-1)} + \mathrm{O}(b^2 P^4). \tag{A.2}$$

Keeping terms linear in bP^2 we rewrite (A.2) as

$$\begin{aligned}L_b\{\mathrm{e}^{aQ^2+bP^2}\} &= \mathrm{e}^{aQ^2} + b \sum_{n=1}^{\infty} \frac{a^{n-1}}{n!} \sum_{m=0}^{n-1} (Q^{2m} P)(P Q^{2(n-m-1)}) \\ &= \mathrm{e}^{aQ^2} + b \sum_{n=1}^{\infty} \frac{a^{n-1}}{n!} \sum_{m=0}^{n-1} (P Q^{2m} + 2\mathrm{i} m Q^{2m-1}) \\ &\quad \times [Q^{2(n-m-1)} P - 2\mathrm{i}(n-m-1) Q^{2(n-m-3/2)}]\end{aligned}$$

where the usual commutation relation of Weyl–Heisenberg algebra, $[Q, P] = \mathrm{i}$ is used.

After some algebraic manipulation, we can write

$$L_b\left\{e^{aQ^2+bP^2}\right\} = Pu_1(Q)P + u_2(Q) \tag{A.3}$$

with

$$u_1(Q) = b\,e^{aQ^2} \qquad u_2(Q) = \left[1 - ab\left(1 + \tfrac{4}{3}aQ^2\right)\right]e^{aQ^2}. \tag{A.4}$$

References

[1] Kulish P P and Reshetikhin N Y 1981 *Zap. Nauchn., Sem. LOMI* **101** 101–10
Kulish P P and Reshetikhin N Y 1983 *J. Sov. Math.* **23** 2435–41
Kulish P P and Sklyanin E K 1982 *Integrable Quantum Field Theories* (*Lecture Notes in Physics* vol 151) ed J Hietarinta (Berlin: Springer) pp 61–119
[2] Faddeev L D 1984 *Recent Advances in Field Theory and Statistical Mechanics (Proc. Les Houches Summer School, Session XXXIX, 1982)* ed J B Zuber and R Stora (Amsterdam: North Holland) pp 561–608
[3] Jimbo M 1985 *Lett. Math. Phys* **10** 63–9
[4] Drinfel'd V G 1985 *Dokl. Akad. Nauk. SSSR* **283** 1060–4
Drinfel'd V G 1985 *Sov. Math.—Dokl.* **32** 254–8
Drinfel'd V G 1987 *Proc. Int. Congress Math. (Berkeley 1986)* vol 1 ed A M Gleason (Providence, RI: American Mathematical Society) pp 798–820
[5] Manin Y I 1987 *Ann. Inst. Fourier* **37** 191–205
[6] Iwata G 1951 *Prog. Theor. Phys.* **6** 524–8
[7] Arik M, Coon D D and Lam Y M 1975 *J. Math. Phys.* **16** 1776–9
Arik M and Coon D D 1976 *J. Math. Phys.* **17** 524–7
[8] Kuryshkin V V 1976 *On Some Generalization of Creation and Annihilation Operators in Quantum Theory (μ-Quantization)* (Moscow: VINITI) deposit N 3936–76 (in Russian)
Kuryshkin V V 1980 *Ann. Fond. L de Broglie* **5** 111–25
[9] Cigler J 1979 *Monat. Math.* **88** 87–105
[10] Jannussis A, Brodimas G, Sourlas D and Zisis V 1981 *Lett. Nuovo Cimento* **30** 123–7
Brodimas G, Jannussis A, Sourlas D, Zisis V and Poulopoulos P 1981 *Lett. Nuovo Cimento* **31** 177–82
[11] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873–8
[12] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581–8
[13] Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415–9
[14] Atakishiev N M and Suslov S K 1990 *Teor. Mat. Fiz.* **85** 64–73
Atakishiev N M and Suslov S K 1990 *Theor. Math. Phys.* **85** 1055–62
[15] Solomon A I and Katriel J 1990 *J. Phys. A: Math. Gen.* **23** L1209–12
Solomon A I and Katriel J 1991 *J. Phys. A: Math. Gen.* **24** 2093–105
[16] Fivel D I 1991 *J. Phys. A: Math. Gen.* **24** 3575–86
[17] Celeghini E, Rasetti M and Vitiello G 1991 *Phys. Rev. Lett.* **66** 2056–9
[18] Chaturvedi S and Srinivasan V 1991 *Phys. Rev.* A **44** 8020–3
Chaturvedi S and Srinivasan V 1991 *Phys. Rev.* A **44** 8024–6
[19] Jannussis A, Brodimas G and Mignani R 1991 *J. Phys. A: Math. Gen.* **24** L775–8
Brodimas G, Jannussis A and Mignani R 1992 *J. Phys. A: Math. Gen.* **25** L329–34
[20] Chiu S-H, Gray R W and Nelson C A 1992 *Phys. Lett.* A **164** 237–42
[21] Ellinas D 1992 *Phys. Rev.* A **45** 3358–61
[22] Man'ko V I and Mendes R V 1993 *Phys. Lett.* A **180** 39–42
[23] Solomon A I and Katriel J 1993 *J. Phys. A: Math. Gen.* **26** 5443–7
McDermott R J and Solomon A I 1994 *J. Phys. A: Math. Gen.* **27** L15–9
[24] Bonatsos D and Daskaloyannis C 1993 *Phys. Lett.* B **307** 100–5
Bonatsos D and Daskaloyannis C 1999 *Prog. Part. Nucl. Phys.* **43** 537–618
[25] Campos R A 1994 *Phys. Lett.* A **184** 173–8
[26] Shanta P, Chaturvedi S, Srinivasan V and Jagannathan R 1994 *J. Phys. A: Math. Gen.* **27** 6433–42
[27] Chung W-S and Klimyk A U 1996 *J. Math. Phys.* **37** 917–32
[28] Katriel J and Quesne C 1996 *J. Math. Phys.* **37** 1650–61
[29] Park S U 1996 *J. Phys. A: Math. Gen.* **29** 3683–96
[30] Isar A and Scheid W 2002 *Physica* A **310** 364–76
Isar A, Sandulescu A and Scheid W 2003 *Physica* A **322** 233–46

[31] Man'ko V I, Marmo G, Solimeno S and Zaccaria F 1993 *Int. J. Mod. Phys.* A **8** 3577–97
Man'ko V I, Marmo G and Zaccaria F 1995 *Phys. Lett.* A **197** 95–9
Manko V I, Marmo G, Sudarshan E C G and Zaccaria F 1997 *Phys. Scr.* **55** 528–41
Man'ko V, Marmo G, Porzio A, Solimeno S and Zaccaria F 2000 *Phys. Rev.* A **62** 053407
[32] Artoni M, Zang J and Birman J L 1993 *Phys. Rev.* A **47** 2555–61
[33] Gruver J L 1999 *Phys. Lett.* A **254** 1–6
[34] Chakrabarti R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** L711–8
[35] Arik M, Demircan E, Turgut T, Ekinci L and Mungan M 1992 *Z. Phys.* C **55** 89–95
[36] Chung W-S, Chung K-S, Nam S-T and Um C-I 1993 *Phys. Lett.* A **183** 363–70
[37] Borzov V V, Damaskinsky E V and Yegorov S B 1995 Some remarks on the representations of the generalized deformed oscillator algebra *Preprint* q-alg/9509022
Borzov V V, Damaskinsky E V and Yegorov S B 1997 *Zap. Nauch. Sem. LOMI* **245** 80–106
Borzov V V, Damaskinsky E V and Yegorov S B 2000 *J. Math. Sci.* **100** 2061
[38] Wigner E P 1951 *Phys. Rev.* **77** 711–2
[39] Streater R F 1967 *Commun. Math. Phys.* **4** 217–36
[40] Brandt R A and Greenberg O W 1969 *J. Math. Phys.* **10** 1168–76
[41] Odaka K, Kishi T and Kamefuchi S 1991 *J. Phys. A: Math. Gen.* **24** L591–6
[42] Quesne C and Vansteenkiste N 1996 *Helv. Phys. Acta* **69** 141–57
[43] Arik M, Atakishiyev N M and Wolf K B 1999 *J. Phys. A: Math. Gen.* **32** L371–6
[44] Feinsilver P 1982 *Rocky Mt. J. Math.* **12** 171–83
Feinsilver P 1987 *Monat. Math.* **104** 89–108
[45] Chaturvedi S, Srinivasan V and Jagannathan R 1993 *Mod. Phys. Lett.* A **8** 3727–34
[46] Jannussis A 1993 *J. Phys. A: Math. Gen.* **26** L233–7
[47] Man'ko V I, Marmo G, Zaccaria F and Sudarshan E C G 1997 *Int. J. Mod. Phys.* B **11** 1281–96
López-Peña R, Man'ko V I and Marmo G 1997 *Phys. Rev.* A **56** 1126
[48] Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer) p 87
[49] Tanás R 2003 *Theory of Classical States of Light* ed V V Dodonov and V I Man'ko (New York: Taylor & Francis) p 277